One class of sign patterns that are potentially eventually exponentially positive

Ber-Lin Yu
Faculty of Mathematics and physics,
Huayin Institute of Technology, Huai'an, Jiangsu, P. R. China

Abstract- A sign pattern is a matrix whose entries belong to the set $\{+,-,0\}$. An $n\times n$ sign pattern $\mathbf{A}$ is said to be potentially eventually exponentially positive (PEEP) if there existssome real matrix A with the same sign pattern as $\mathbf{A}$ and some nonnegative integer t_0 such that for all $t\geq t_0$,

$$e^{kA}=\sum_{k=0}^{+\infty}\frac{t^k}{k!}A^k>0.$$

Identifying the necessary and sufficient conditions for an $n\times n$ ($n\geq 4$) sign pattern to be PEEP and classifying all PEEP sign patterns are two open problems. In this short paper, we investigate the potential eventual exponential positivity of one class of special sign patterns $\mathbf{A}$ of order 6 whose underlying graph $G(\mathbf{A})$ is obtained from a path graph of order 5 by adding a pendent vertex. It is shown that for the special class of sign patterns, $\mathbf{A}$ is PEEP if and only if all the nonzero off-diagonal entries of $\mathbf{A}$ is positive, which consequently classifies all its PEEP sign patterns of the specific form.

Keywords –potentially eventually exponentially positive sign pattern, minimal potentially eventually exponentially positive sign pattern, potentially eventually positive sign pattern

I. Introduction

A *sign pattern* is a matrix $\mathbf{A}=(\alpha_{ij})$ with entries in the set $\{+,-,0\}$. An $n\times n$ real matrix A with the same sign pattern as $\mathbf{A}$ is called a *realization* of $\mathbf{A}$. The set of all realizations of sign pattern $\mathbf{A}$ is called the *qualitative class* of $\mathbf{A}$ and is denoted by $Q(\mathbf{A})$. An $n\times n$ sign pattern $\mathbf{A}$ is said to be *potentially eventually positive* (*PEP* for short), if there exists some $A\in Q(\mathbf{A})$ such that A is eventually positive; see, e.g., [2]. An $n\times n$ sign pattern $\mathbf{A}$ is said to be *potentially eventually exponentially positive* (*PEEP*, for short), if there exists some real matrix $A\in Q(\mathbf{A})$ and some nonnegative integer t_0 such that for all $t\geq t_0$,

$$e^{kA}=\sum_{k=0}^{+\infty}\frac{t^k}{k!}A^k>0.$$

Sign patterns that allow an eventually exponentially positive matrix have been studied first in [1], where some sufficient conditions and some necessary conditions for an $n\times n$ sign pattern to be PEEP have been established. However, how to characterize the $n\times n$ PEEP sign patterns when $n\geq 4$ is a open problem.

In this short note, we focus on one class of specific tree sign patterns $\mathbf{A}$ with the underlying graph $G(\mathbf{A})$ being obtained from the path graph P_5 by adding a pendent vertex. It is shown that for the class of special sign patterns, $\mathbf{A}$ is PEEP if and only if all the nonzero off-diagonal entries of $\mathbf{A}$ are positive, which classifies all PEEP sign patterns and identifies the exactly one minimal PEEP sign pattern.

II. MAIN RESULTS

For an $n \times n$ sign pattern $\mathbf{A} = (\alpha_{ij})$, the *positive part* of $\mathbf{A}$ is defined to be $\mathbf{A}^{+} = (\alpha_{ij}{}^{+})$, where $(\alpha_{ij}{}^{+}) = +$ for $\alpha_{ij} = +$, otherwise $(\alpha_{ij}{}^{+}) = 0$. In [1], it has been shown that if sign pattern $\mathbf{A}^{+}$ is irreducible, then $\mathbf{A}$ is PEEP. Furthermore, since every superpattern of PEEP sign patterns is also PEEP, it is sufficient to determine whether each (or some) subpattern of a given sign pattern is PEEP or not. Motivated by this, we call an $n \times n$ sign pattern $\mathbf{A}$ a minimal PEEP sign pattern, if $\mathbf{A}$ is PEEP and no proper subpattern of $\mathbf{A}$ is PEEP.

Following [1], let $\mathbf{A}_{D(+)}$ (respectively, $\mathbf{A}_{D(0)}$ and $\mathbf{A}_{D(-)}$) be the sign pattern obtained from sign pattern $\mathbf{A}$ by changing all diagonal entries to $+$ (respectively, 0 and $-$). The following lemma is cited from [1] for the reader's convenience.

Lemma 1. *If an $n \times n$ sign pattern $\mathbf{A}$ is PEEP, then $\mathbf{A}_{D(+)}$ is PEP.*

Now we turn to focus on the class of specific tree sign patterns $\mathbf{A}$ whose underlying graph is obtained from the path graph of order 5 by adding a pendent vertex and is shown in Figure 1.

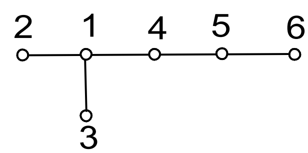

Figure 1. The graph of sign patterns

Up to equivalence, without loss of generality, let sign pattern

$$\mathbf{A} = (\alpha_{ij}) = \begin{pmatrix} ? & * & * & * & 0 & 0 \\ * & ? & 0 & 0 & 0 & 0 \\ * & 0 & ? & 0 & 0 & 0 \\ * & 0 & 0 & ? & * & 0 \\ 0 & 0 & 0 & * & ? & * \\ 0 & 0 & 0 & 0 & * & ? \end{pmatrix} \tag{1}$$

where ? denotes an entry from $\{+, -, 0\}$ and * denotes a nonzero entry. Note that $\mathbf{A}$ is a combinatorially symmetric sign pattern and the graph $G(\mathbf{A})$ is a tree.

It is necessarily mentioned that the potential eventual positivity of $\mathbf{A}$ has been investigated in [3]. For the potential eventual exponential positivity of $\mathbf{A}$, we cite Proposition 12 in [3] as Lemma 2 to establish the necessary and sufficient conditions for the sign pattern in the class of sign patterns of the form (1) to be PEEP.

Lemma 2. *If the tree sign pattern $\mathbf{A}$ of the form (1) is PEP, then the nonzero off-diagonal entries of $\mathbf{A}$ are all positive, that is to say, $\alpha_{12} = \alpha_{21} = +$, $\alpha_{13} = \alpha_{31} = +$, $\alpha_{14} = \alpha_{41} = +$, $\alpha_{45} = \alpha_{54} = +$ and $\alpha_{56} = \alpha_{65} = +$.*

Theorem 1. *If the tree sign pattern $\mathbf{A}$ of the form (1) is PEEP, then $\alpha_{1i} = \alpha_{i1} = +$ for $i = 2, 3, 4$, and $\alpha_{1,i+1} = \alpha_{i+1,1} = +$ for $i = 4, 5$.*

Proof. Since tree sign pattern $\mathbf{A}$ of the form (1) is PEEP, $\mathbf{A}_{D(+)}$ is PEP by Lemma 1. By Lemma 2, all nonzero off-diagonal entries of $\mathbf{A}_{D(+)}$ must be $+$. It follows that $\alpha_{1i} = \alpha_{i1} = +$ for $i = 2, 3, 4$, and $\alpha_{1,i+1} = \alpha_{i+1,1} = +$ for $i = 4, 5$.

Now we identify the minimal PEEP sign patterns of the form (1).

Theorem 2. *Up to equivalence, the sign pattern*

$$\mathbf{A}^{\circ} = (\alpha_{ij}^{\circ}) = \begin{pmatrix} 0 & + & + & + & 0 & 0 \\ + & 0 & 0 & 0 & 0 & 0 \\ + & 0 & 0 & 0 & 0 & 0 \\ + & 0 & 0 & 0 & + & 0 \\ 0 & 0 & 0 & + & 0 & + \\ 0 & 0 & 0 & 0 & + & 0 \end{pmatrix}$$

is the unique minimal PEEP sign pattern in the class of sign patterns of the form (1).

Proof. Since the positive part $(\mathbf{A}^{\circ})^{+}$ is irreducible, $\mathbf{A}^{\circ}$ is PEEP. Every proper subpattern of $\mathbf{A}^{\circ}$ is not irreducible, and thus is not PEEP. It follows that $\mathbf{A}^{\circ}$ is a minimal PEEP sign pattern. For the uniqueness, assume B is an another minimal PEEP sign pattern of the form $\mathbf{A}$. Then by Lemma 2, all nonzero off-diagonal entries of B must be $+$. If B has at least one nonzero diagonal entry, then B must be a proper superpattern of $\mathbf{A}^{\circ}$, and hence B is not minimal PEEP. Consequently, all diagonal entries of B must be 0. It follows that B is equivalent to $\mathbf{A}^{\circ}$.

Note that Theorem 2 indicates that for the class of sign patterns $\mathbf{A}$ of the form (1), there exists exactly one minimal PEEP sign pattern. The following Theorem 3 classifies all the PEEP sign patterns in the class of the sign patterns of the form (1).

Theorem 3. *Let $\mathbf{A}$ be the sign pattern of the form (1). Then $\mathbf{A}$ is PEEP if and only if $\mathbf{A}$ is equivalent to a superpattern of $\mathbf{A}^{\circ}$.*

Proof. If $\mathbf{A}$ is equivalent to a superpattern of $\mathbf{A}^{\circ}$, then $\mathbf{A}$ by PEEP by Theorem 2 and the fact that every superpattern of a PEEP sign pattern is also PEEP. For the necessity, if $\mathbf{A}$ is minimal PEEP, then $\mathbf{A}$ is equivalent to $\mathbf{A}^{\circ}$ by Theorem2. Furthermore if $\mathbf{A}$ is not minimal PEEP, then $\mathbf{A}$ is a proper superpattern of the exactly one minimal PEEP sign pattern $\mathbf{A}^{\circ}$.

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